JOURNAL OF DIFFERENTIAL EQUATIONS **84**, 1–14 (1990)

# An Homology Index Generalizing Fuller's Index for Periodic Orbits

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*Communicated by Jack K. Hale*

Received October 9, 1987; revised February 10, 1989

The homology Fuller index is defined. It is defined for compact sets of periodic orbits that are isolated in the phase-cross-period space for a vector field on a smooth manifold. The index is related to the Fuller index, and the motivation for the definition comes from homology constructions used by Fuller in the work on his index. A continuation theory for the index is presented. The relationship between the homology Fuller index and the Fuller index is established. © 1990 Academic Press, Inc.

## INTRODUCTION

Fuller [1] defines an index for isolated sets of periodic orbits of a vector field on a smooth manifold $M$. The index is defined for periodic orbits, not as orbits in the phase space $M$, but as orbits in phase-cross-period space $M \times \mathbb{R}^+$. Thus, an index is associated to each multiple of an isolated periodic orbit of the vector field.

For an isolated periodic orbit (with multiplicity $m$) in $M \times \mathbb{R}^+$ the index is defined to be equal to the fixed-point index of the $m$th iterate of a Poincaré map for the orbit in $M$ divided by $m$. For isolated compact sets of periodic orbits, the index is defined to be the sum of the indices of each orbit in a finite collection of periodic orbits obtained by continuation of the original set under perturbation of the vector field.

Fuller proves that the index is additive and invariant under continuation of the vector field. Of course, these properties are necessary for the index to be well defined. To show the invariance under continuation, Fuller associates a homology class to the index of each isolated compact set of periodic orbits and then appeals to simple properties of homology theory.

To avoid assigning trivial homology classes to sets of periodic orbits that have nontrivial index (e.g., when $M$ is simply connected), Fuller associates a sequence of manifolds $M^k$, $k$ prime, with vector fields $X^k$ on $M^k$ having



0022-0396/90 $3.00

periodic sets that naturally correspond to the original, and he shows that for $k$ sufficiently large, the homology class associated to the corresponding set of periodic orbits is nontrivial.

Our approach is to define the index immediately for all isolated compact sets of periodic orbits. This is done by using naturally defined sequences of homology and cohomology maps. The definition of these sequences is motivated by the constructions of Fuller. With our approach the additivity and invariance under continuation of the index are simple consequences of the definition, and the homology information associated to periodic orbits is retained in the index.

The continuation theory that we present for the homology Fuller index is modelled after Conley's continuation theory in [2]. The idea is to define a space of isolated compact sets of periodic orbits for a continuous family of vector fields on $M$ and to give this space a topology that makes it a sheaf over the parameter space. As a result, a topological structure associated to the global continuation and bifurcation framework of the periodic orbits is defined. The continuation theorem then states that the index is invariant on connected components of the space of periodic orbits.

The development of the homology Fuller index and the continuation theory are presented in Sections 1 and 2, respectively. In Section 3 we exhibit the relationship between the homology Fuller index and the Fuller index and prove that the homology Fuller index contains more information than the Fuller index.

In [3] Chow and Mallet-Paret use the Fuller index to prove a global Hopf bifurcation theorem. In our setting similar results may be possible by taking advantage of the more general homology index and the topological structure of the space of isolated compact sets of periodic orbits for a continuous family of vector fields. We leave that problem for further investigation.

## 1. The Homology Fuller Index

Assume throughout that $M$ is a $C^\infty$ $n$-manifold with $n \geqslant 2$. Let $X$ be a $C^\infty$ vector field on $M$. For $x \in M$ we denote (the elements of) the trajectory through $x$ by $x \cdot t$.

The setting for the index theory is the phase-cross-period space $M \times \mathbb{R}^+$. Specifically, a point $(x, p) \in M \times \mathbb{R}^+$ is called a periodic point of the vector field if $x \cdot p = x$. Note that if $x \cdot 0 = x$, then $(x, p)$ is a periodic point for all $p$. For such $x$ we also call the point $(x, p)$ a stationary point of the vector field. If $(x, p)$ is a periodic point, then so is $(x \cdot t, p)$ for all $t$. Let $(x, p)$ be a periodic point that is not a stationary point; then we call the set $\pi = \{(x \cdot t, p) \mid t \in \mathbb{R}\}$ a periodic orbit (with period $p$) of the vector field, and if $q$ is the minimal positive number such that $x \cdot q = x$, then we say that $\pi$ has minimal period $q$ and multiplicity $m = p/q$.

A union of periodic orbits is called a periodic set. A periodic set is said to be isolated if it is the maximal collection of periodic points contained in a compact neighborhood $N$ of itself in $M \times \mathbb{R}^+$. Under such circumstances the neighborhood $N$ is called an isolating neighborhood for the vector field. If $N$ is also a manifold (with boundary), then $N$ is called an isolating manifold neighborhood.

Note that if $(x, p)$ is a stationary point, then it is for all $p$. Therefore no isolated periodic set (and no isolating neighborhood) contains stationary points.

PROPOSITION 1.1. *If $C$ is an isolated periodic set, then $C$ is compact and there is an upper bound for the multiplicities of the periodic orbits in $C$.*

*Proof.* Since $C$ is contained in a compact neighborhood of itself, it is enough to show that $C$ is closed. Thus, suppose that $(x, p)$ is the limit of a sequence $\{(x_i, p_i)\} \subset C$. Then $x_i$ converges to $x$, and $x_i \cdot p_i$ converges to $x \cdot p$. Since $x_i \cdot p_i = x_i$, it follows that $x \cdot p = x$; therefore $x \in C$.

Now suppose there is no bound for the multiplicities of the periodic orbits in $C$. Then there is a sequence of points $\{(x_i, p_i)\} \subset C$ such that $(x_i, p_i)$ is in a periodic orbit $\pi_i$ with multiplicity greater than $i$. Since the set $\{p_i\}$ is bounded, the minimal periods of the periodic orbits $\pi_i$ tend to 0. Therefore a limit point of the sequence is a stationary point contained in $C$, contradicting the fact that $C$ contains no stationary points. ∎

If $C$ is an isolated periodic set, then we let $m(C)$ denote the supremum of the multiplicities of the periodic orbits in $C$.

If $N$ is an isolating neighborhood with maximal periodic set $C$, then there exists an isolating manifold neighborhood $N'$ of $C$ contained in the interior of $N$. This can be shown by covering $C$ with a sufficiently fine polyhedral set (which is possible since $M$ is $C^\infty$, therefore triangulable) and then letting $N'$ be a regular neighborhood (which is necessarily a manifold) of the polyhedral set (see [4, 5]).

Let $\Delta$ denote the diagonal in the product of a set with itself. If $N$ is an isolating neighborhood for $X$, and $N_0 \subset N$ is disjoint from the maximal periodic set in $N$, then a map

$$g: (N, N_0) \to (M \times M, M \times M \backslash \Delta)$$
$$g(x, t) = (x, x \cdot t)$$

is defined. The map $g$ is called a coincidence map for the vector field.

Let $H_*(\cdot)$ and $H^*(\cdot)$ denote singular homology and cohomology, respectively, with integral coefficients.

The first step in defining the homology Fuller index is to define the following homology index for isolated periodic sets of $X$. We consider the cases where $M$ is orientable and nonorientable separately. To begin, assume that $M$ is orientable. Let $C$ be an isolated periodic set of $X$, and let $N$ be an isolating manifold neighborhood of $C$. Fix an orientation of $M$, and let $M \times \mathbb{R}^+$ have the orientation induced by this orientation of $M$ and the standard orientation of $\mathbb{R}^+$. Associated to these orientations there are fundamental classes $O \in H^n(M \times M,\ M \times M \backslash \Delta)$ and $O^N \in H_{n+1}(N, \partial N)$. Define $I(C)$, the homology index of $C$, to be the image of $O$ under the sequence of maps

$$H^n(M \times M, M \times M \backslash \Delta) \xrightarrow{g^*} H^n(N, \partial N) \xrightarrow{\cap O^N} H_1(N)$$

$$\xrightarrow{i_*} H_1(M \times \mathbb{R}^+) \xrightarrow{p_*} H_1(M).$$

$g^*$ is the cohomology map induced by a coincidence map $g$ for $X$, $\cap O^N$ is the homomorphism defined by cap product with $O^N$, and $i_*$ and $p_*$ are the homology maps induced by inclusion and projection, respectively.

In Proposition 1.2 below, we show that $I(C)$ is well defined.

Roughly speaking, the sequence computes the fixed-point index of a Poincaré map for the orbits (via the map $g^*$) and associates it to the homology carried by the orbits (via $\cap O^N$, $i_*$, and $p_*$). Geometrically, this is best illustrated by considering an isolated periodic orbit. We give a cursory explanation of these relationships here; a rigorous treatment is presented in Section 3, where we establish the relationship between the homology Fuller index and the Fuller index.

Consider Fig. 1. $\pi$ is an isolated multiplicity-one period-$p$ periodic orbit in $M \times \mathbb{R}^+$, and $x \in \pi$. Let $B$ be a disk transverse to $\pi$ in $M$ at $x$, and, with $\tilde{B} \subset B$, assume $P$: $\tilde{B} \to B$ is a Poincaré map. Let $B\varepsilon = B \cdot [-\varepsilon, \varepsilon]$ (identified appropriately with $B \times [-\varepsilon, \varepsilon]$). Given $\alpha < \varepsilon$, let $\tilde{B}\alpha = \tilde{B} \times [p - \alpha, p + \alpha]$ (identified appropriately with $\tilde{B} \times [-\alpha, \alpha]$). Assume $\tilde{B}\alpha$ represents a generator, $\gamma$, for $H^n(N, \partial N)$, where $N$ is an isolating neighborhood of $\pi$ in $M \times \mathbb{R}^+$.

The map on $\tilde{B}\alpha$ defined by $c(y, s) = (y, 0, P(y), s)$ locally represents $g$. On cohomology, $c^*$ precisely represents $g^*$. With an appropriate choice of orientation classes, it can then be seen that $g^*$ maps $O$ to the product of $\gamma$ and the fixed-point index of $P$ (the latter being, roughly, the intersection number of the graph of $P$ with the diagonal in $B \times B$). $\cap O^N$ maps $\gamma$ to its dual homology class in $N$; that class is represented by the periodic orbit $\pi$. It follows that if $\pi$ is an isolated multiplicity-one periodic orbit, then $I(\pi)$ is the product of the fixed-point index of a Poincaré map for the orbit and the homology in $M$ represented by the orbit.

We now show that $I(C)$ is well defined.

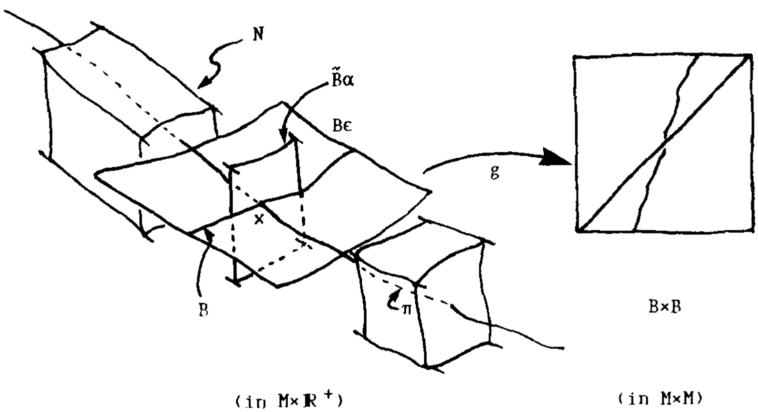


FIG. 1. Schematic local representation of the coincidence map $g$.

PROPOSITION 1.2. *The class $I(C) \in H_1(M)$ is independent of the choice of orientation of $M$ and of the choice of isolating manifold neighborhood $N$ of $C$.*

*Proof.* With a different choice of orientation of $M$ the classes $O$ and $O^N$ change sign. Clearly this results in no change in $I(C)$.

Now assume that $N_i$, $i = 1, 2$, are isolating manifold neighborhoods of $C$. Let $L$ be an isolating manifold neighborhood of $C$ such that $L \subset \operatorname{int}(N_i)$ for each $i$. Consider the following diagram for $i = 1, 2$.

$$\begin{tikzcd}
 & H^n(N_i, \partial N_i) \arrow[r] & H_1(N_i) \arrow[dr, "i_*"] & \\
H^n(M \times M, M \times M \backslash \Delta) \arrow[ur, "g^*"] \arrow[r, "g^*"] \arrow[dr, "g^*"'] & H^n(N_i, N_i \backslash \operatorname{int} L) \arrow[u, "i^*"'] \arrow[d, "i^*", "\approx"'] \arrow[r, "\cap i_*^{-1}(O^L)"] & H_1(N_i) \arrow[u, equal] \arrow[r, "i_*"] & H_1(M \times \mathbb{R}^+) \\
 & H^n(L, \partial L) \arrow[r, "\cap O^L"] & H_1(L) \arrow[u, "i_*"'] \arrow[ur, "i_*"'] &
\end{tikzcd}$$

The maps $i^*$ and $i_*$ are induced by inclusion. By excision, the homology and cohomology maps induced by inclusion between the spaces $(L, \partial L)$ and $(N_i, N_i \backslash \operatorname{int} L)$ are isomorphisms. Thus, the vertical map on the lower left is an isomorphism, and the cap product homomorphism $\cap i_*^{-1}(O^L)$ is defined. Since the diagram is commutative for $i = 1, 2$, it follows that $I(C)$ is defined independent of the choice of the isolating neighborhood $N$. ∎

For the case where $M$ is nonorientable, let $M_0$ be a tubular neighborhood of $M$ considered as a submanifold of some Euclidean space. $M_0$ can be coordinatized by $(x, v)$, where $x \in M$ and $v$ is normal to $M$ at $x$. Extend the vector field $X$ to a vector field $X_0$ on $M_0$ via the definition $X_0(x, v) = X(x) - v$. An isolated periodic set $C$ of $X$ is an isolated periodic set $C_0$ of $X_0$. Since $M_0$ is orientable, $I(C_0)$ is defined. Let $I(C)$ be the image of $I(C_0)$ under the map $H_1(M_0) \to H_1(M)$ induced by the map $(x, v) \to x$.

We leave it to the reader to confirm that if $M$ is nonorientable, then $I(C)$, as defined, depends on neither the embedding of $M$ into a Euclidean space nor the particular tubular neighborhood of $M$ (see [4, 6]).

It is not difficult to see that the homology index, $I$, satisfies all of the desirable properties of an index: nontriviality implies the existence of periodic orbits, it is additive, and it is invariant under continuation. However, the index may be undesirably trivial; this occurs, for example, when $M$ is simply connected. To overcome such trivialities, the homology index is extended to the homology Fuller index below.

Associated to $M$ there is a sequence $\{M^k\}$ of $kn$-manifolds, $k$ prime, defined as follows. Let $M'_k = M \times \cdots \times M$ be the product of $M$, $k$ times. Set $M''_k = \{(x_1, ..., x_k) \in M'_k \mid x_i \neq x_j \ \forall\ i \neq j\}$. $M''_k$ is a connected $C^\infty$ $nk$-manifold. $\mathbb{Z}_k$ acts on $M''_k$ with an action generated by $z: M''_k \to M''_k$, where $z(x_1, ..., x_k) = (x_2, ..., x_k, x_1)$. Set $M^k$ equal to the quotient of $M''_k$ under this action; i.e., $M^k = M''_k/\mathbb{Z}_k$. $M^k$ is a connected $C^\infty$ $nk$-manifold. Denote the class of $(x_1, ..., x_k)$ in $M^k$ by $[x_1, ..., x_k]$.

A vector field $X$ on $M$ induces a vector field $X^k$ on $M^k$ with trajectories $[x_1, ..., x_k] \cdot t = [x_1 \cdot t, ..., x_k \cdot t]$.

Let $(x, p)$ be a periodic point of $X$ contained in $\pi$, a periodic orbit having multiplicity $m$. For $k > m$ set

$$x''_k = (x, x \cdot p/k, ..., x \cdot (k-1)\, p/k) \in M''_k,$$

and denote $[x''_k] \in M^k$ by $x^k$. $(x^k, p/k)$ is a periodic point of $X^k$ contained in $\pi^k$, a periodic orbit which also has multiplicity $m$. If $C$ is a periodic set of $X$, and $k > m(C)$, then set $C^k$ equal to the union of the $\pi^k$ for $\pi \subset C$, and call this set the $k$-orbits associated to $C$.

PROPOSITION 1.3. *If $C$ is an isolated periodic set of $X$, then $C^k$ is an isolated periodic set of $X^k$.*

*Proof.* We claim that $C^k$ is compact. To see this, let $\{x^k_n\}$ be a sequence in $C^k$. The sequence $\{x_n\} \subset C$ has a convergent subsequence, $\{x_{n_i}\}$, converging in $C$. It follows that the sequence $\{x^k_{n_i}\}$ converges in $C^k$; therefore $C^k$ is compact. Now, given that $C^k$ is compact, one can similarly show that $C^k$ is isolated. ∎

*Note.* Many of the results discussed herein can be established via limit arguments similar to those in Propositions 1.1 and 1.3. In the remainder of the paper we leave such details to the reader.

The homology Fuller index of an isolated periodic set is an element of the module obtained from the collection of the $H_1(M^k)$, $k$ prime, by forming the quotient of the direct product over the direct sum; i.e., the index is a tail of a sequence, defined over prime $k$, of classes in $H_1(M^k)$. Thus, define

$$\mathscr{H}(M) = \bigotimes_{k \text{ prime}} H_1(M^k) \Big/ \bigoplus_{k \text{ prime}} H_1(M^k).$$

DEFINITION 1.4. If $C$ is an isolated periodic set of $X$, then we define $I_H(C)$, the homology Fuller index of $C$, to be the class $[I^k(C)] \in \mathscr{H}(M)$, where $I^k(C)$ is trivial for $k \leqslant m(C)$, and $I^k(C) = I(C^k)$ for $k > m(C)$.

Note that the homology Fuller index clearly satisfies the additivity property (i.e., if $C_1$ and $C_2$ are disjoint isolated periodic sets for $X$, then $I_H(C_1 \cup C_2) = I_H(C_1) + I_H(C_2)$) and is nontrivial only when $C$ is nonempty. The invariance under continuation of the index follows from the fact that a continuous change in the vector field results in a continuous change in the coincidence map, which then remains unchanged on the cohomology level. This is made precise in the next section where we present the continuation theory for the homology Fuller index.

## 2. CONTINUATION

Let $X_\lambda$ be a continuous family of vector fields on $M$ defined for $\lambda \in \Lambda$, an interval in $\mathbb{R}$. Let $\mathscr{P}$, the space of isolated periodic sets of the family of vector fields, be the set of pairs of the form $(C_\lambda, \lambda)$, where $C_\lambda$ is an isolated periodic set of the vector field $X_\lambda$.

For each compact $N \subset M$, let $\Lambda(N)$ be the set of $\lambda \in \Lambda$ such that $N$ is an isolating neighborhood for the vector field $X_\lambda$. Define $\sigma_N\colon \Lambda(N) \to \mathscr{P}$ by $\sigma_N(\lambda) = (C_\lambda, \lambda)$, where $C_\lambda$ is the maximal periodic set in $N$ for the vector field $X_\lambda$. $\mathscr{P}$ is topologized via a basis $\mathscr{B}$ consisting of sets of the form $\sigma_N(U)$, where $N \subset M$ is compact and $U \subset \Lambda$ is open. To show that $\mathscr{B}$ is a basis for a topology on $\mathscr{P}$, the following proposition is needed.

PROPOSITION 2.1. *Let $N$, $N' \subset X$ be compact, and assume $\mu \in \Lambda(N) \cap \Lambda(N')$. If $N$ and $N'$ have the same maximal periodic set of the vector field $X_\mu$, then there exists an open neighborhood $W$ of $\mu$ in $\Lambda(N) \cap \Lambda(N')$ such that, for each $\lambda \in W$, the maximal periodic sets of the vector field $X_\lambda$ in $N$ and $N'$ are equal.*

*Proof.* $N'' = (N \cup N') \backslash (\mathrm{int}(N \cap N'))$ contains no periodic points of the vector field $X_\mu$. We claim that there exists an open neighborhood $W$ of $\mu$ such that for each $\lambda \in W$, $N''$ contains no periodic points of the vector field $X_\lambda$. Assume not, then there exist a sequence $\{\lambda_i\}$ converging to $\mu$ and a collection $\{(x_i, p_i)\} \subset N''$ of periodic points of the vector field $X_{\lambda_i}$. An accumulation point of the collection $\{(x_i, p_i)\}$ is a periodic point of the vector field $X_\mu$ in $N''$, a contradiction. Clearly $W \subset \Lambda(N) \cap \Lambda(N')$, and for the vector fields $X_\lambda$ with $\lambda \in W$, the maximal periodic sets in $N$ and $N'$ are equal. ∎

An immediate consequence of Proposition 2.1 is that $\Lambda(N)$ is open for each compact $N \subset X$. Note furthermore that the collection $\mathscr{B}$, described above, is a basis for a topology on $\mathscr{P}$; i.e., $\mathscr{B}$ clearly covers $\mathscr{P}$, and if $(C_\lambda, \lambda) \in \sigma_{N_1}(U_1) \cap \sigma_{N_2}(U_2)$, then there is an open neighborhood $V$ of $\lambda$ contained in the intersection of $U_1$, $U_2$, $\Lambda(N_1)$, $\Lambda(N_2)$, and $\Lambda(N_1 \cap N_2)$ such that $\sigma_{N_1 \cap N_2}(V) \subset \sigma_{N_1}(U_1) \cap \sigma_{N_2}(U_2)$.

Define $\pi: \mathscr{P} \to \Lambda$ by $\pi(C_\lambda, \lambda) = \lambda$. It is easy to see that $\pi$ is a surjective local homeomorphism and that for each compact $N \subset X$, $\pi \mid \sigma_N(\Lambda(N))$ is a homeomorphism with inverse $\sigma_N$. Note that $\mathscr{P}$ is locally path connected, and therefore the path components and connected components of $\mathscr{P}$ coincide.

Definition 2.2. *For $i = 0, 1$, assume $\lambda_i \in \Lambda$ and $C_{\lambda_i}$ is an isolated periodic set of the vector field $X_{\lambda_i}$. We say that $C_{\lambda_0}$ and $C_{\lambda_1}$ are related by continuation if $(C_{\lambda_0}, \lambda_0)$ and $(C_{\lambda_1}, \lambda_1)$ lie in the same component of $\mathscr{P}$.*

The following proposition is the main homology Fuller index continuation result.

Proposition 2.3. *If $C_{\lambda_0}$ and $C_{\lambda_1}$ are related by continuation, then $I_{\mathrm{H}}(C_{\lambda_0}) = I_{\mathrm{H}}(C_{\lambda_1})$.*

*Proof.* Let $(C_{\lambda_s}, \lambda_s)$, $0 \leqslant s \leqslant 1$, be a path in $\mathscr{P}$ connecting $(C_{\lambda_0}, \lambda_0)$ and $(C_{\lambda_1}, \lambda_1)$. It is easy to see that there is a bound $m$ (independent of $s$) for the multiplicities of the orbits in $C_{\lambda_s}$, $0 \leqslant s \leqslant 1$. We show that $I^k(C_{\lambda_0}) = I^k(C_{\lambda_1})$ for $k > m$ and $M^k$ orientable; the nonorientable case follows similarly.

Let $X_s^k$ be the vector field on $M^k$ induced by $X_{\lambda_s}$, $C_s^k$ be the $k$-orbits associated to $C_{\lambda_s}$, and $N_s$ be an isolating manifold neighborhood of $C_s^k$. For each $s$, we claim there exists an open neighborhood $U_s$ in $[0, 1]$ such that $\jmath \in U_s$ implies that $N_s$ is an isolating neighborhood of $C_\jmath^k$. To prove the claim, first note that Proposition 2.1 implies there is an open neighborhood

$U_s''$ of $s$ in $[0, 1]$ such that $N_s$ is an isolating neighborhood for the vector field $X_\varsigma^k$ for each $\varsigma \in U_s''$. Then, via appropriate limit arguments, one can show that there exists a neighborhood $U_s'$ of $s$ in $U_s''$ such that $C_\varsigma^k \subset N_s$ for each $\varsigma \in U_s'$, and, finally, that there exists a neighborhood $U_s$ of $s$ in $U_s'$ such that $C_\varsigma^k$ is the maximal periodic set in $N_s$ for each $\varsigma \in U_s$.

Now, since $N_s$ is an isolating neighborhood for the vector field $X_\varsigma^k$ for each $\varsigma \in U_s$, the coincidence map for the vector field $X_\varsigma^k$,

$$g_\varsigma\colon (N_s, \partial N_s) \to (M^k \times M^k, M^k \times M^k \backslash \Delta),$$

is defined for each $\varsigma \in U_s$. Furthermore, since $g_\varsigma$ varies continuously with $\varsigma$, the cohomology map

$$g_\varsigma^*\colon H^*(M^k \times M^k, M^k \times M^k \backslash \Delta) \to H^*(N_s, \partial N_s)$$

is independent of $\varsigma$.

Therefore, for $\varsigma \in U_s$, $I^k(C_{\lambda_\varsigma})$ is independent of $\varsigma$. It now easily follows that $I^k(C_{\lambda_0}) = I^k(C_{\lambda_1})$, and thus $I_{\mathrm{H}}(C_{\lambda_0}) = I_{\mathrm{H}}(C_{\lambda_1})$. ∎

## 3. The Relationship between the Homology Fuller Index and the Fuller Index

In this section we describe the relationship between the homology Fuller index and the Fuller index of an isolated periodic set of a vector field $X$ on $M$.

The Fuller index of an isolated multiplicity-$m$ periodic orbit $\pi$ is defined to be the fixed-point index of the $m$th iterate of the Poincaré map of the orbit in $M$ divided by $m$. For an isolated periodic set $C$, the Fuller index is defined to be the sum of the indices of each orbit in a finite collection of isolated periodic orbits obtained by continuation of $C$ under perturbation of the vector field. Thus, the Fuller index of an isolated periodic set $C$ is a rational number; we denote it by $I_{\mathrm{F}}(C)$.

We start by showing that there is a naturally defined homomorphism between $\mathscr{H}(M)$ and the rationals that maps the homology Fuller index to the Fuller index.

To begin, note that the identification mapping of $M_k''$ to $M^k$ is a regular covering whose group of covering transformations equals $\mathbb{Z}_k$. We identify the covering transformation $z$, defined above, with $1 \in \mathbb{Z}_k$. By covering space theory (see [7]) there is a homomorphism $\mu_k$ mapping the fundamental group of $M^k$ to $\mathbb{Z}_k$, and since $\mathbb{Z}_k$ is abelian, this can be realized as a homomorphism $\mu_k\colon H_1(M^k) \to \mathbb{Z}_k$. Define

$$\mathscr{Z}\mathscr{K} = \bigotimes_{k\,\text{prime}} \mathbb{Z}_k \Big/ \bigoplus_{k\,\text{prime}} \mathbb{Z}_k.$$

The maps $\mu_k$, above, induce a homomorphism $\mu\colon \mathscr{H}(M) \to \mathscr{Z}\mathscr{K}$.

The rationals are contained naturally in $\mathscr{Z}\mathscr{K}$ via an identification $p/q = [r_2, r_3, r_5, r_7, ..., r_k, ...]$, where $r_k = 0$ if $k \leqslant |p|$, $|q|$, and $r_k = p/q \pmod{k}$ (i.e., $p = qr_k \pmod{k}$) if $k > |p|$, $|q|$. We leave it to the reader to confirm that this is a well-defined identification of the rationals with a field in $\mathscr{Z}\mathscr{K}$.

Now, viewing the rationals as contained in $\mathscr{Z}\mathscr{K}$, we show that if $C$ is an isolated periodic set, then the homomorphism $\mu$ maps the homology Fuller index of $C$ to the Fuller index of $C$; i.e., $\mu(I_{\mathrm{H}}(C)) = I_{\mathrm{F}}(C)$.

Since both indices satisfy the additivity property and are invariant under continuation, and since we can perturb the vector field so that an isolated periodic set continues to a finite collection of isolated periodic orbits (see [8]), it suffices to show that if $\pi$ is an isolated periodic orbit, then $\mu(I_{\mathrm{H}}(\pi)) = I_{\mathrm{F}}(\pi)$.

Thus, assume $\pi$ is an isolated periodic orbit of $X$ having multiplicity $m$. Denote by $\hat{\pi}$ the corresponding orbit in $M$. If $\pi \subset W \subset M \times \mathbb{R}^+$, then let $\lambda(\pi) \in H_1(W)$ denote the homology class corresponding to the path $\{(x \cdot t, p) | (x, p) \in \pi,\ 0 \leqslant t \leqslant p/m\}$. Define $\lambda(\hat{\pi}) \in H_1(M)$ to be the projection of $\lambda(\pi)$. If $k > m$, then for $\pi^k$, the $k$-orbit associated to $\pi$, define $\hat{\pi}^k$, $\lambda(\pi^k)$, and $\lambda(\hat{\pi}^k)$ similarly. Note that in each of these cases the homology class under consideration represents one cycle of the corresponding orbit. Let $P$, $P_k$ be the Poincaré maps for $\hat{\pi}$, $\hat{\pi}^k$, respectively, and $P^m$, $P_k^m$ be their $m$th iterates. $i_{\mathrm{fp}}$ is used to denote the fixed-point indices of these maps.

Via the following propositions, the desired result on the map $\mu$ is established.

**Proposition 3.1.** $\mu_k(\lambda(\hat{\pi}^k)) = 1/m \pmod{k}$.

*Proof.* Assume $x \in \hat{\pi}$, and let $x_k'' \in M_k''$ and $x^k \in \hat{\pi}^k$ be defined as in Section 1. Set $s(\hat{\pi}^k) \in H_1(M^k)$ equal to the homolgy class corresponding to the path $\{x^k \cdot t \,|\, 0 \leqslant t \leqslant p/k\}$. Note that the lift of this path to a path in $M_k''$ starting at $x_k''$ ends at $z(x_k'')$. Therefore, by covering space theory, it follows that $\mu_k(s(\hat{\pi}^k)) = 1$. Furthermore, note that $s(\hat{\pi}^k) = m\lambda(\hat{\pi}^k)$, and therefore $\mu_k(\lambda(\hat{\pi}^k)) = 1/m \pmod{k}$. ∎

Note that Proposition 3.1 implies that the orbit $\hat{\pi}^k$ represents a nontrivial homology class in $H_1(M^k)$.

PROPOSITION 3.2. $I(\pi) = i_{\mathrm{fp}}(P^m)\, \lambda(\hat{\pi})$.

*Proof.* We prove this for the case where $M$ is orientable. Using the easily derived equality between the fixed-point index of the Poincaré map of an isolated periodic orbit of $X$ in nonorientable $M$ and the fixed-point index of the Poincaré map of the corresponding orbit in the extended vector field on orientable $M_0$, the case for $M$ nonorientable then follows.

Let $B$ denote the closed unit ball in $R^{n-1}$. If $\tilde{B} \subset B$ and $\varepsilon > 0$, then denote the set $\tilde{B} \times [-\varepsilon, \varepsilon] \subset R^n$ by $\tilde{B}\varepsilon$. Let $N$ be an isolating manifold neighborhood of $\pi$ which is also the closure of a tubular neighborhood of $\pi$. The inclusion map $i\colon (N, \partial N) \to (N, N\backslash\pi)$ induces homology and cohomology isomorphisms.

Refer to Fig. 1 in Section 1. Let $U \subset M$ be a neighborhood of $x \in \hat{\pi}$ coordinatized by $B2\varepsilon$ such that:

(1) $(0, 0) = x$,

(2) if $0 < s, t < \varepsilon$ and $(y, s) \in U$, then $(y, s) \cdot t = (y, s + t)$,

(3) there exist $\tilde{B} \subset B$ and $P^m\colon \tilde{B} \to B$, a represention for the $m$th iterate of the Poincaré map for $\hat{\pi}$, such that if $y \in \tilde{B}$, then $(y, 0) \cdot p = (P^m(y), 0)$,

(4) there exists $\alpha \in (0, \varepsilon)$ such that $U \times [p - \alpha, p + \alpha] \subset N$.

That (2) can be satisfied by a neighborhood $U$ of $x$ is insured by the existence of a flow box containing $x$, and property (3) can be satisfied by a reparameterization of the flow so that the transversal section through $x$ is an invariant section (see [8]).

Assume that the orientation of $M$ is induced by the standard orientation of $B2\varepsilon$. Let $O \in H^n(M \times M, M \times M\backslash\Delta)$ and $O^N \in H^{n+1}(N, \partial N) \approx H^{n+1}(N, N\backslash\pi)$ denote the corresponding fundamental classes.

Define $j\colon (\tilde{B}\alpha, \tilde{B}\alpha\backslash 0) \to (N, N\backslash\pi)$ by $j(y, s) = (y, 0, s + p)$. $j$ maps $\tilde{B}\alpha$ to the product of a section in $M$ transverse to $\hat{\pi}$ at $x$ with an interval $[p - \alpha, p + \alpha]$ in period space $\mathbb{R}^+$. Let $O^{\tilde{B}\alpha} \in H^n(\tilde{B}\alpha, \tilde{B}\alpha\backslash 0)$ be the fundamental class corresponding to the standard orientation of $\tilde{B}\alpha$. $j^*$, the cohomology map induced by $j$, is an isomorphism in dimension $n$, and by the established conventions on orientations, it follows that $(j^*)^{-1}$ maps $O^{\tilde{B}\alpha}$ to a class $\gamma \in H^n(N, N\backslash\pi)$ satisfying $\gamma \cap O^N = \lambda(\pi) \in H_1(N)$.

Define $c\colon (\tilde{B}\alpha, \tilde{B}\alpha\backslash 0) \to (B\varepsilon \times B\varepsilon, B\varepsilon \times B\varepsilon\backslash\Delta)$ by $c(y, s) = (y, 0, P^m(y), s)$; $c$ is a representation for the restriction of the coincidence map $g$ to $\tilde{B}\alpha \times \{0\}$. Let $O^{B\varepsilon} \in H^n(B\varepsilon \times B\varepsilon,\ B\varepsilon \times B\varepsilon\backslash\Delta)$ be the fundamental class corresponding to the standard orientation of $B\varepsilon$. From fixed-point theory (see [9]) it follows that $c^*(O^{B\varepsilon}) = \kappa O^{\tilde{B}\alpha}$, where $\kappa$ is the coincidence index of the maps $\mathrm{id} \times 0$, $P^m \times \mathrm{id}\colon \tilde{B} \times [-\alpha, \alpha] \to B \times [-\varepsilon, \varepsilon]$ (with $0(s) = 0$). Using the multiplicative property of the coincidence index, it is easily seen that $\kappa = i_{\mathrm{fp}}(P^m)$.

Now consider the diagram

$$\begin{tikzcd}
H^n(M\times M, M\times M\backslash\Delta) \arrow[r, "g^*"] \arrow[dr, "g^*"] \arrow[dd, "i^*", "\approx"'] & H^n(N, \partial N) \arrow[r, "\cap O^N"] & H_1(N) \\
 & H^n(N, N\backslash\pi) \arrow[u, "i^*", "\approx"'] \arrow[ur, "\cap O^N"] \arrow[d, "j^*", "\approx"'] & \\
H^n(B\varepsilon\times B\varepsilon, B\varepsilon\times B\varepsilon\backslash\Delta) \arrow[r, "c^*"] & H^n(\tilde{B}\alpha, \tilde{B}\alpha\backslash 0) &
\end{tikzcd}$$

The diagram commutes, and by the above remarks it follows that $g^*(O)\cap O^N = i_{\mathrm{fp}}(P^m)\,\lambda(\pi)$. To each side of this equation, apply the composition of projection onto $H_1(M)$ with inclusion into $H_1(M\times\mathbb{R}^+)$; the result is $I(\pi) = i_{\mathrm{fp}}(P^m)\,\lambda(\tilde{\pi})$. ∎

Using basic properties from fixed-point theory Fuller shows that for $k>m$, $i_{\mathrm{fp}}(P_k^m) = i_{\mathrm{fp}}(P^m)$. Therefore, with the aid of Proposition 3.2, we have that $I^k(\pi) = i_{\mathrm{fp}}(P^m)\,\lambda(\tilde{\pi}^k)$ for $k>m$. Combining this with Proposition 3.1, it follows that $\mu_k(I^k(\pi)) = i_{\mathrm{fp}}(P^m)/m \pmod{k}$ for $k>m$, implying that $\mu(I_{\mathrm{H}}(\pi)) = I_{\mathrm{F}}(\pi)$. Thus, piecing together the above results, the proof of the following relationship between the homology Fuller index and the Fuller index is immediate.

THEOREM 3.3. *If $C$ is an isolated periodic set of $X$, then $\mu(I_{\mathrm{H}}(C)) = I_{\mathrm{F}}(C)$.*

Thus, via the map $\mu$, one sees that the homology Fuller index contains the information in the Fuller index. In Theorem 3.5 below, we show that the homology Fuller index actually contains more information than the Fuller index. This follows because the homology Fuller index retains homology information that is not present in the Fuller index.

For example, while the Fuller indices of the multiplicity-one attracting periodic orbits on the Möbius band in Fig. 2 are equal, the homology Fuller indices are not.

In general, if two periodic orbits represent nonhomologous cycles on a manifold $M$, then one cannot be continued to the other. If the orbits are nondegenerate, then this information is present in the homology Fuller index but not necessarily in the Fuller index. To prove this, it is necessary to show that if two periodic orbits represent nonhomologous cycles, then, for $k$ large, so do their associated $k$-orbits. This fact is a consequence of

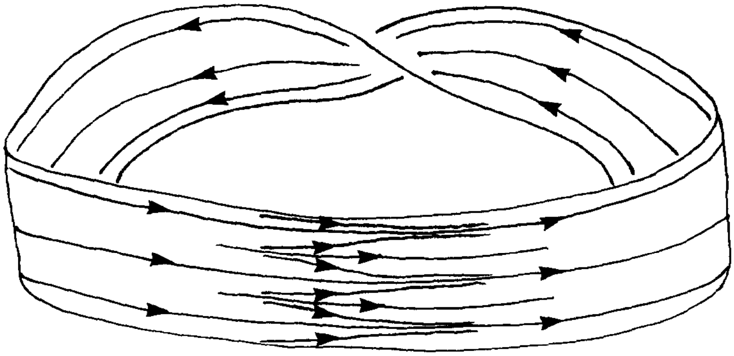

FIG. 2. The multiplicity-one attracting periodic orbits on the Möbius band have equal Fuller indices but distinct homology Fuller indices.

PROPOSITION 3.4. *Assume $\pi_1$ and $\pi_2$ are periodic orbits of $X$ having multiplicities $m_1$ and $m_2$, respectively, and are such that $m_1\lambda(\hat{\pi}_1) \neq m_2\lambda(\hat{\pi}_2)$. Then if $k > \max\{m_1, m_2\}$, then $m_1\lambda(\hat{\pi}_1^k) \neq m_2\lambda(\hat{\pi}_2^k)$.*

*Proof.* We prove the contrapositive; i.e., if $m_1\lambda(\hat{\pi}_1^k) = m_2\lambda(\hat{\pi}_2^k)$, then $m_1\lambda(\hat{\pi}_1) = m_2\lambda(\hat{\pi}_2)$. For $i = 1, 2$, assume $\pi_i$ has period $p_i$ and $x_i \in \hat{\pi}_i$. For $j = 1, ..., k$ and $t \in [0, p_i/k]$ let $\alpha_{i,j}(t) = x_i \cdot (j-1)\, p_i/k \cdot t$. Clearly the cycle $\sum_j \alpha_{i,j}$ represents $m_i\lambda(\hat{\pi}_i)$. Furthermore, the loop $\alpha_i(t) := [\alpha_{i,1}(t), ..., \alpha_{i,k}(t)]$, $t \in [0, p_i/k]$, in $M^k$ represents $m_i\lambda(\hat{\pi}_i^k)$.

With $x_i^k$ defined as above, let $\beta_0$ be a path in $M^k$ from $x_1^k$ to $x_2^k$ and let $\beta_0^{-1}$ be the corresponding return path. Since $m_1\lambda(\hat{\pi}_1^k) = m_2\lambda(\hat{\pi}_2^k)$, it follows that the loop $\alpha_1\beta_0\alpha_2^{-1}\beta_0^{-1}$ is homologically trivial and therefore represents an element, $\beta_1\beta_2\beta_1^{-1}\beta_2^{-1}\cdots\beta_{2n-1}\beta_{2n}\beta_{2n-1}^{-1}\beta_{2n}^{-1}$, of the commutator subgroup of the fundamental group of $M^k$. Thus there is a map $\omega$ from a $(4n+4)$-gon, $D$, to $M^k$ with boundary equal to

$$\alpha_1 + \beta_0 - \alpha_2 - \beta_0 - \beta_1 - \beta_2 + \beta_1 + \beta_2 - \cdots - \beta_{2n-1} - \beta_{2n} + \beta_{2n-1} + \beta_{2n}.$$

Now, $\omega$ lifts to a map from $D$ to $M_k'' \subset M \times \cdots \times M$. The lift can be regarded as a map $(\omega_1, ..., \omega_k)$: $D \to M \times \cdots \times M$, and the sum $\sum_j \omega_j$ represents a 2-chain in $M$. We claim that the boundary of $\sum_j \omega_j$ is $\sum_j \alpha_{1,j} - \sum_j \alpha_{2,j}$, implying that $m_1\lambda(\hat{\pi}_1) = m_2\lambda(\hat{\pi}_2)$.

To prove the claim, first note that the map $(\alpha_{i,1}, ..., \alpha_{i,k})$ is a lift of $\alpha_i$ to $M_k''$, and any other lift of $\alpha_i$ to $M_k''$ is obtained by a cyclic permutation of the $\alpha_{i,j}$. A lift of the path $\beta_i$ to $M_k''$ can be regarded as a map $(\beta_{i,1}, ..., \beta_{i,k})$, where each $\beta_{i,j}$ is a path in $M$, and any other lift of $\beta_i$ is obtained by a cyclic permutation of the $\beta_{i,j}$. If $\beta_{i,j}^{-1}$ is the return path corresponding to

$\beta_{i,j}$, then $(\beta_{i,1}^{-1}, \ldots, \beta_{i,k}^{-1})$ is a lift of $\beta_i^{-1}$, and the other lifts are obtained, as above, by cyclic permutations. Now it is clear that for each $\omega_j$,

$$\partial(\omega_j) = \alpha_{1,\sigma_1(j)} + \beta_{0,\sigma_2(j)} - \alpha_{2,\sigma_3(j)} - \beta_{0,\sigma_4(j)} - \beta_{1,\sigma_5(j)} + \cdots + \beta_{2n,\sigma_{4n+4}(j)},$$

where each $\sigma_i$ is a cyclic permutation of $\{1, \ldots, k\}$. It follows that

$$\partial\left(\sum_j \omega_j\right) = \sum_j \alpha_{1,j} - \sum_j \alpha_{2,j}. \quad \blacksquare$$

By Theorem 3.3, nonequality of Fuller indices implies nonequality of homology Fuller indices. We finish this paper by showing that if two isolated nondegenerate periodic orbits have equal Fuller indices but represent nonhomologous cycles, then their homology Fuller indices are distinct. In this way the homology Fuller index contains more information than the Fuller index.

THEOREM 3.5. *Assume $\pi_1$ and $\pi_2$ are isolated periodic orbits of $X$ having multiplicities $m_1$ and $m_2$, respectively. If $I_{\mathrm{F}}(\pi_1) = I_{\mathrm{F}}(\pi_2) \neq 0$, and $m_1\lambda(\hat{\pi}_1) \neq m_2\lambda(\hat{\pi}_2)$, then $I_{\mathrm{H}}(\pi_1) \neq I_{\mathrm{H}}(\pi_2)$.*

*Proof.* Assume $k > \max\{m_1, m_2\}$; then since $m_1\lambda(\hat{\pi}_1) \neq m_2\lambda(\hat{\pi}_2)$, it follows from Proposition 3.4 that $m_1\lambda(\hat{\pi}_1^k) \neq m_2\lambda(\hat{\pi}_2^k)$. For $i = 1, 2$, let $F_i$ be the fixed-point index of the $m_i$th iterate of the Poincaré map for $\hat{\pi}_i$. Then $F_1/m_1 = F_2/m_2 \neq 0$, implying $F_1\lambda(\hat{\pi}_1^k) \neq F_2\lambda(\hat{\pi}_2^k)$. Therefore, $I^k(\pi_1) \neq I^k(\pi_2)$ for large $k$, and it follows that $I_{\mathrm{H}}(\pi_1) \neq I_{\mathrm{H}}(\pi_2)$. $\blacksquare$

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